

LOCAL COHOMOLOGICAL PROPERTIES OF HOMOGENEOUS ANR COMPACTA

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ABSTRACT. In accordance with the Bing-Borsuk conjecture [2], we show that if X is an n -dimensional homogeneous metric *ANR* continuum and $x \in X$, then there is a local basis at x consisting of connected open sets U such that the cohomological properties of \overline{U} and bdU are similar to the properties of the closed ball $\mathbb{B}^n \subset \mathbb{R}^n$ and its boundary \mathbb{S}^{n-1} . We also prove that a metric *ANR* compactum X of dimension n is dimensionally full-valued if and only if the group $H_n(X, X \setminus x; \mathbb{Z})$ is not trivial for some $x \in X$, where \mathbb{Z} is the group of integers. This implies that every 3-dimensional homogeneous metric *ANR* compactum is dimensionally full-valued.

1. INTRODUCTION

The Bing-Borsuk conjecture [2] asserts that a homogeneous Euclidean neighborhood retract is a topological manifold. In accordance with that conjecture, we show that the local cohomological structure of any n -dimensional homogeneous metric *ANR* continuum is similar to the local structure of \mathbb{R}^n , see Theorem 1.1 below. We also establish conditions for a metric *ANR* compactum X to satisfy the equality $\dim(X \times Y) = \dim X + \dim Y$ for all compact metric spaces Y (any such X is said to be *dimensionally full-valued*). It follows from these conditions that every 3-dimensional homogeneous *ANR* compactum is dimensionally full-valued (Corollary 1.5), thus providing a partial answer to one of the problems accompanying the Bing-Borsuk conjecture (whether homogeneous metric *ANR*'s are dimensionally full-valued).

Everywhere in this paper by a space we mean a homogeneous metric *ANR* continuum X with $\dim_G X = n$, where $n \geq 2$ and G is a fixed countable abelian group or a principal ideal domain (PID) with unity. Reduced Čech homology $H_n(X; G)$ and cohomology groups $H^n(X; G)$ with coefficient from G are considered everywhere below. Let us recall that for any abelian group G the cohomology groups $H^n(X; G)$, $n \geq 2$, are isomorphic to the groups $[X, K(G, n)]$ of pointed homotopy classes of maps from X to $K(G, n)$, where $K(G, n)$ is the Eilenberg-MacLane space of type (G, n) , see [22]. The cohomological dimension $\dim_G X$ is the largest integer m such

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that there exists a closed set $A \subset X$ with $H^m(X, A; G) \neq 0$. Equivalently, $\dim_G X \leq n$ iff every map $f: A \rightarrow K(G, n)$ can be extended to a map $\tilde{f}: X \rightarrow K(G, n)$.

Suppose (K, A) is a pair of closed subsets of a space X with $A \subset K$. Following [2], we say that K is an n -homology membrane spanned on A for an element $\gamma \in H_n(A; G)$ provided γ is homologous to zero in K , but not homologous to zero in any proper closed subset of K containing A . Similarly, K is said to be an n -cohomology membrane spanned on A for an element $\gamma \in H^n(A; G)$ if γ is not extendable over K , but it is extendable over every proper closed subset of K containing A . Here, $\gamma \in H^n(A; G)$ is not extendable over K means that γ is not contained in the image $j_{K,A}^n(H^n(K; G))$, where $j_{K,A}^n: H^n(K; G) \rightarrow H^n(A; G)$ is the homomorphism generated by the inclusion $A \hookrightarrow K$.

We note the following simple fact, which will be used in this paper and follows from Zorn's lemma and the continuity of Čech cohomology [22]: *If A is a closed subset of a compact space X and γ is an element of $H^n(A; G)$ not extendable over X , then there exists an n -cohomology membrane for γ spanned on A .*

We also say that a closed set $A \subset X$ is a cohomological carrier of a non-zero element $\alpha \in H^n(A; G)$ if $j_{A,B}^n(\alpha) = 0$ for every proper closed subset $B \subset A$. If $H^n(A; G) \neq 0$, but $H^n(B; G) = 0$ for every closed proper subset $B \subset A$, then A is called an (n, G) -bubble.

Theorem 1.1. *Let X be a homogeneous metric ANR continuum with $\dim_G X = n$, where G is a countable PID with unity and $n \geq 2$. Then every point x of X has a basis \mathcal{B}_x of open sets $U \subset X$ satisfying the following conditions:*

- (1) $\text{int} \overline{U} = U$ and the complement of $\text{bd} U$ has two components, one of which is U ;
- (2) $H^{n-1}(\overline{U}; G) = 0$ and \overline{U} is an $(n-1)$ -cohomology membrane spanned on $\text{bd} U$ for any non-zero $\gamma \in H^{n-1}(\text{bd} U; G)$;
- (3) $\text{bd} U$ is an $(n-1, G)$ -bubble and $H^{n-1}(\text{bd} U; G)$ is a finitely generated G -module.

The restriction $n \geq 2$ in Theorem 1.1 is needed because of Lemma 2.8, which is used in the proof of Theorem 1.1.

Remark. Condition (1) from Theorem 1.1 implies that $\dim_G \text{bd} U = n-1$, see [13].

Theorem 1.2. *Let X be as in Theorem 1.1 and G be a countable group. If a closed subset $K \subset X$ is an $(n-1)$ -cohomology membrane spanned on A for some closed set $A \subset K$ and some $\gamma \in H^{n-1}(A; G)$, then $(K \setminus A) \cap \overline{X \setminus K} = \emptyset$.*

Corollary 1.3. *In the setting of Theorem 1.2, if $U \subset X$ is open and $f: U \rightarrow X$ is an injective map, then $f(U)$ is open in X .*

We already mentioned that a compactum X is dimensionally full-valued if $\dim(X \times Y) = \dim X + \dim Y$ for any compact metric space Y , or equivalently, $\dim_G X = \dim_{\mathbb{Z}} X$ for any abelian group G . Recent work of Bryant [5] was believed to provide a positive answer to the question whether any homogeneous metric ANR is dimensionally full-valued, but Bryant discovered a gap in the proof of one of the theorems from [5]. The question whether $\dim(X \times Y) = \dim X + \dim Y$ if both X and Y are homogeneous compact ANRs was raised in [6] and [10]. Theorem 1.4 below provides some necessary and sufficient conditions for ANR spaces to be dimensionally full-valued.

Theorem 1.4. *The following conditions are equivalent for any metric ANR-compactum X of dimension $\dim X = n$:*

- (1) X is dimensionally full-valued;
- (2) There is a point $x \in X$ with $H_n(X, X \setminus x; \mathbb{Z}) \neq 0$;
- (3) $\dim_{\mathbb{S}^1} X = n$.

Corollary 1.5. *Every homogeneous metric ANR compactum X with $\dim X = 3$ is dimensionally full-valued.*

2. SOME PRELIMINARY RESULTS

In this section, if not stated otherwise, G is a countable abelian group and X denotes a homogeneous metric ANR continuum with $\dim_G X = n$, $n \geq 2$. If $H^n(X; G) \neq 0$, then $H^n(B; G) = 0$ for all proper closed subsets B of X , see [23]. Obviously, this is true when $H^n(X; G) = 0$. Therefore, all proper closed subsets of X have trivial n -cohomology groups.

We begin with the following analogue of Theorem 8.1 from [2] (it is here that the countability of G is used).

Proposition 2.1. *Theorem 1.2 holds under the additional assumption that K is contractible in a proper subset of X .*

Proof. According to the duality between homology and cohomology for countable groups [12, viii 4G)], for any compact metric space Y the groups $H_{n-1}(Y, G^*)$ and $H^{n-1}(Y; G)^*$ are isomorphic, where G^* and $H^{n-1}(Y; G)^*$ denote the character groups of G and $H^{n-1}(Y; G)$, respectively. Here both $H^{n-1}(Y; G)$ and G are considered as discrete groups. Using this duality, we can show that K is an $(n-1)$ -homology membrane for some $\beta \in H_{n-1}(A, G^*)$ spanned on A . Indeed, consider the homomorphism $j_{K,A}^{n-1} : H^{n-1}(K; G) \rightarrow H^{n-1}(A; G)$. Since γ is not extendable over K , $\gamma \notin G_A = j_{K,A}^{n-1}(H^{n-1}(K; G))$. Considering $H^{n-1}(A; G)$ as a discrete group, we can find a character $\beta : H^{n-1}(A; G) \rightarrow \mathbb{S}^1$ such that $\beta(\gamma) \neq e$ and $\beta(G_A) = e$, where e is the unit of \mathbb{S}^1 . On the other hand, γ is extendable over every proper closed subset B of K which contains A . Therefore, γ is contained in the image of $j_{B,A}^{n-1} : H^{n-1}(B; G) \rightarrow H^{n-1}(A; G)$ for any such B . Then the composition $j_{K,A}^{n-1} \circ \beta$ is the trivial character of $H^{n-1}(K; G)$, while the composition $j_{B,A}^{n-1} \circ \beta$ is non-trivial for any proper closed subset B of K containing A . So, β is homologous to zero in K , but not homologous to zero in any proper closed

subset of K containing A . Hence, K is an $(n-1)$ -homology membrane for β spanned on A .

Now, assume that $(K \setminus A) \cap \overline{X \setminus K} \neq \emptyset$. Then following the proof of Theorem 16.1 from [3] (see also [2, Theorem 8.1]), we can find a proper closed subset Γ of X and a non-zero element $\alpha \in H_n(\Gamma, G^*)$. This means that $H^n(\Gamma; G) \neq 0$, a contradiction. \square

Since the Bing-Borsuk result used in the proof of Proposition 2.1 was established for locally homogeneous spaces, Proposition 2.1 remains valid for locally homogeneous spaces X such that $H^n(A; G)$ is trivial for any proper closed subset $A \subset X$.

Corollary 2.2. *Let $A \subset X$ be a closed subset and K an $(n-1)$ -cohomology membrane for some $\gamma \in H^{n-1}(A; G)$ spanned on A . Then $K \setminus A$ is connected. If, in addition, K is contractible in a proper subset of X , then $K \setminus A$ is an open subset of X .*

Proof. Suppose $K \setminus A$ is the union of two non-empty, disjoint open sets U and V . Then $K \setminus U$ and $K \setminus V$ are closed proper subsets of K such that $(K \setminus U) \cap (K \setminus V) \subset A$. Hence, γ is extendable over each of these sets and, because A contains their common part, γ is extendable over K . The last conclusion contradicts the fact that K is $(n-1)$ -cohomology membrane for γ .

If K is contractible in a proper subset of X , then $(K \setminus A) \cap \overline{X \setminus K} = \emptyset$ (see Proposition 2.1). Hence, $K \setminus A$ is open in X . \square

Corollary 2.3. *For any closed set $Z \subset X$ one has $\dim_G Z = n$ if and only if Z has a non-empty interior in X .*

Proof. This was established by Seidel in [19] for the covering dimension. His arguments can be modified for \dim_G . If $\dim_G Z = n$, we may assume that Z is contractible in a proper subset of X (this can be done because X is locally contractible and \dim_G satisfies the countable sum theorem). Since $\dim_G Z = n$, there exists a closed set $A \subset Z$ such that $H^n(Z, A; G) \neq 0$. On the other hand, $H^n(Z; G) = 0$ (as a proper closed subset of X). So, according to the exact sequence

$$H^{n-1}(Z; G) \xrightarrow{j_{Z,A}^{n-1}} H^{n-1}(A; G) \xrightarrow{\delta} H^n(Z, A; G) \rightarrow 0$$

there exists $\gamma \in H^{n-1}(A; G)$ not extendable over Z . Hence, as it was noted above, we can find a closed subset K of Z such that K is an $(n-1)$ -cohomological membrane for γ spanned on A . So, $K \setminus A$ is open in X (by Corollary 2.2) and $K \setminus A \subset Z$.

If Z has a non-empty interior, then it contains an open set U in X with $\dim_G U = n$. So, $\dim_G Z = n$. \square

Lemma 2.4. *Let a closed set $F \subset X$ with $H^{n-1}(F; G) \neq 0$ be contractible in an open set $U \subset X$. If \overline{U} is contractible in a proper subset of X , then F separates \overline{W} for any open set $W \subset X$ containing U .*

Proof. Indeed, there is a closed set P in X such that $P \subset U$ and F is contractible in P . Then any non-zero element $\gamma \in H^{n-1}(F; G)$ is not extendable over P (otherwise γ , considered as a map from F to $K(G, n-1)$, would be homotopic to a constant because F is contractible in P). This yields the existence of an $(n-1)$ -cohomology membrane $K_\gamma \subset P$ for γ spanned on F . Because \overline{U} is contractible in a proper subset of X , so is K_γ . Hence, by Proposition 2.1, $(K_\gamma \setminus F) \cap \overline{X \setminus K_\gamma} = \emptyset$. The last equality implies that F separates any \overline{W} such that $W \subset X$ is open and contains U . \square

Lemma 2.5. *Suppose $U \subset X$ is open and $P \subsetneq X$ is closed such that $\overline{U} \subsetneq P$ and $H^{n-1}(bdU; G)$ contains elements not extendable over \overline{U} . Then, there exists $\gamma \in H^{n-1}(bdU; G) \setminus L$ extendable over $P \setminus V$, where $V = \text{int}(\overline{U})$ and $L = j_{\overline{U}, bdU}^{n-1}(H^{n-1}(\overline{U}; G))$. Moreover, if $L = 0$, then every $\gamma \in H^{n-1}(bdU; G)$ is extendable over $P \setminus V$.*

Proof. Indeed, since $H^{n-1}(bdU; G)$ contains elements not extendable over \overline{U} , L is a proper subgroup of $H^{n-1}(bdU; G)$. Consider the homomorphism $j_{P \setminus V, bdU}^{n-1}: H^{n-1}(P \setminus V; G) \rightarrow H^{n-1}(bdU; G)$. It suffices to show that the image of $H^{n-1}(P \setminus V; G)$ under $j_{P \setminus V, bdU}^{n-1}$ is not contained in L . To this end, suppose $j_{P \setminus V, bdU}^{n-1}(H^{n-1}(P \setminus V; G)) \subset L$. Consider the Mayer-Vietoris exact sequence, where $A = P \setminus V$ and $\varphi(\gamma_1, \gamma_2) = j_{A, bdU}^{n-1}(\gamma_2) - j_{\overline{U}, bdU}^{n-1}(\gamma_1)$ for $\gamma_1 \in H^{n-1}(\overline{U}; G)$, $\gamma_2 \in H^{n-1}(A; G)$:

$$H^{n-1}(\overline{U}; G) \oplus H^{n-1}(A; G) \xrightarrow{\varphi} H^{n-1}(bdU; G) \xrightarrow{\Delta} H^n(P; G) \rightarrow$$

Obviously, $L_U = \varphi(H^{n-1}(\overline{U}; G) \oplus H^{n-1}(A; G)) \subset L$. Consequently, any $\gamma \in H^{n-1}(bdU; G) \setminus L$ is not contained in L_U . Hence, $\Delta(\gamma) \neq 0$ for all $\gamma \in H^{n-1}(bdU; G) \setminus L$. So, $H^n(P; G) \neq 0$, a contradiction (recall that the n -th cohomology groups of all proper closed sets in X are trivial).

If $L = 0$, then $j_{\overline{U}, bdU}^{n-1}(\gamma_1) = 0$ for all $\gamma_1 \in H^{n-1}(\overline{U}; G)$, so $\varphi(\gamma_1, \gamma_2) = j_{A, bdU}^{n-1}(\gamma_2)$. Since $\Delta(H^{n-1}(bdU; G)) = 0$, we obtain that for any element $\gamma \in H^{n-1}(bdU; G)$ there exist $\gamma_1 \in H^{n-1}(\overline{U}; G)$ and $\gamma_2 \in H^{n-1}(A; G)$ such that $\varphi(\gamma_1, \gamma_2) = \gamma$. Hence, $\gamma = j_{A, bdU}^{n-1}(\gamma_2)$, which means that γ is extendable over A . This completes the proof. \square

Lemma 2.6. *If $U \subset X$ is a connected open set and \overline{U} is contractible in a proper subset of X , then \overline{U} is an $(n-1)$ -cohomology membrane spanned on bdU for every $\gamma \in H^{n-1}(bdU; G)$ not extendable over \overline{U} .*

Proof. Observe first that U is dense in $V = \text{int}(\overline{U})$, so V is also connected. Let γ be an element of $H^{n-1}(bdU; G)$ not extendable over \overline{U} . Then there exists a closed subset $K \subset \overline{U}$ such that K is an $(n-1)$ -cohomology membrane for γ spanned on bdU . Since K is contractible in a proper subset of X (as a subset of \overline{U}), by Proposition 2.1, $(K \setminus bdU) \cap \overline{X \setminus K} = \emptyset$. Hence, $K \setminus bdU$ is open in X . This implies that $K = \overline{U}$, otherwise V would be the union of the non-empty disjoint open sets $V \setminus K$ and $(K \setminus bdU) \cap V$. Therefore, \overline{U} is an $(n-1)$ -cohomology membrane spanned on bdU for γ . \square

The last two statements of this section (Lemma 2.7 and Lemma 2.8) hold for arbitrary compactum X .

Lemma 2.7. *Let X be an arbitrary compactum and $A \subset X$ be a carrier for a non-zero element $\gamma \in H^{n-1}(A; G)$ with $\dim_G A \leq n-1$, $n \geq 2$. Then A is connected.*

Proof. Suppose A is not connected, so A is the union of two closed disjoint non-empty sets A_1 and A_2 . Then $H^{n-1}(A; G)$ is isomorphic to the direct sum $H^{n-1}(A_1; G) \oplus H^{n-1}(A_2; G)$ and γ is identified with the pair (γ_1, γ_2) , where $\gamma_i = j_{A, A_i}^{n-1}(\gamma)$, $i = 1, 2$. Because A is a carrier of γ and A_i are proper closed non-empty subsets of A , $\gamma_1 = \gamma_2 = 0$. So, $\gamma = 0$, a contradiction. \square

Since that $\dim_G A = 0$ is equivalent with $\dim A = 0$, Lemma 2.8 is not valid for $n = 1$. For example, if A consists of two different points, then there exists a non-trivial element of $\gamma \in H^0(A; \mathbb{Z})$ such that A is a carrier of γ .

Suppose G is a group (resp., a ring). Let $F \subset Z \subset X$ be compact sets. We say that F is an $(n-1, G)$ -bubble with respect to a subgroup (resp., a submodule) $L \subset H^{n-1}(Z; G)$ if the group (resp., the submodule) $j_{Z, F}^{n-1}(L) \subset H^{n-1}(F; G)$ is non-trivial, but $j_{Z, B}^{n-1}(L) \subset H^{n-1}(B; G)$ is trivial for any closed proper subset $B \subset F$.

Lemma 2.8. *Let G be a group (resp., a ring). If Z is a closed subset of an arbitrary compactum X and $L \subset H^{n-1}(Z; G)$ is a non-trivial and finitely generated subgroup (resp., a submodule), then Z contains a non-empty closed subset F such that F is an $(n-1, G)$ -bubble with respect to L .*

Proof. If L has one generator γ , we just take a closed set $F \subset Z$, which is a carrier for γ . Then $\beta = j_{Z, F}^{n-1}(\gamma)$ and $\beta_B = j_{Z, B}^{n-1}(\gamma)$ are generators, respectively, of $j_{Z, F}^{n-1}(L) \subset H^{n-1}(F; G)$ and $j_{Z, B}^{n-1}(L) \subset H^{n-1}(B; G)$ for any closed set $B \subset Z$. So, $j_{Z, B}^{n-1}(L) = 0$ for every proper closed subset B of F because $j_{Z, B}^{n-1}(\gamma) = j_{F, B}^{n-1}(\beta) = 0$. Hence, F is an $(n-1, G)$ -bubble with respect to L . Suppose our lemma is true for any such set Z and a subgroup (resp., a submodule) $L \subset H^{n-1}(Z; G)$ with $\leq k$ generators. In case L has $k+1$ generators $\gamma_1, \dots, \gamma_k, \gamma_{k+1}$, we first take a closed non-empty set $F_1 \subset Z$, which is a carrier for γ_1 . So, $j_{Z, B}^{n-1}(\gamma_1) = 0$ for any proper closed subset B of F_1 . If $H^{n-1}(B; G) = 0$ for all closed $B \subsetneq F_1$, then F_1 is as required. If $j_{Z, B^*}^{n-1}(L) \neq 0$ for some closed proper set $B^* \subset F_1$, then $j_{Z, B^*}^{n-1}(L)$ is generated by the set $\{j_{Z, B^*}^{n-1}(\gamma_i) : i = 2, 3, \dots, k+1\}$. According to our inductive assumption, there exists a closed non-empty set $F \subset B^*$ being an $(n-1, G)$ -bubble in B^* with respect to $j_{Z, B^*}^{n-1}(L)$. Then F is an $(n-1, G)$ -bubble in Z with respect to L . \square

3. PROOF OF THEOREMS 1.1, 1.2 AND COROLLARY 1.3

In this section X continues to be as in Section 2, but G is assumed to be a countable PID (the last condition is used in the proof of Claim 1).

Proof of Theorem 1.1. As in the proof of Proposition 2.1, we may suppose that X is connected and $H^n(C; G) = 0$ for any closed proper subset C of X .

Moreover, we equip X with a convex metric d generating its topology (such a metric exists, see [1]). According to [16, Theorem 2], there exists a closed subset $Y \subset X$ with $\dim_G Y = n$ and a dense open subset D of Y satisfying the following property: any $y \in D$ has sufficiently small neighborhoods U_y in Y such that the homomorphism $j_{\overline{U}_y, bd_Y \overline{U}_y}^{n-1}$ is not surjective (here $bd_Y \overline{U}_y$ denotes the boundary of \overline{U}_y in Y). Because Y has a non-empty interior in X (by Corollary 2.3), there exists a point $x \in \text{int}(Y) \cap D$, a connected open neighborhood W_x of x in X and an element $\alpha_x \in H^{n-1}(bd \overline{W}_x; G)$ such that α_x is not extendable over \overline{W}_x . We can suppose that \overline{W}_x is contractible in a proper subset of X . So, by Lemma 2.6, \overline{W}_x is an $(n-1)$ -cohomology membrane for α_x spanned on $bd \overline{W}_x$. Because X is homogeneous, it suffices to construct the required base \mathcal{B}_x at that particular point x . We define \mathcal{B}'_x to be the family of all open connected subsets $U \subset X$ containing x such that $U = \text{int}(\overline{U})$ and \overline{U} is contractible in W_x . Then \mathcal{B}'_x is a local base at x and $bdU = bd \overline{U}$ for all $U \in \mathcal{B}'_x$.

Claim 1. Every $U \in \mathcal{B}'_x$ has the following properties:

- (i) \overline{U} is an $(n-1)$ -cohomology membrane for some element of the group $H^{n-1}(bdU; G)$;
- (ii) the module $L_U = j_{\overline{W}_x \setminus U, bdU}^{n-1}(H^{n-1}(\overline{W}_x \setminus U; G)) \subset H^{n-1}(bdU; G)$ is non-trivial and finitely generated;
- (iii) the module $H^{n-1}(bdU; G)$ is finitely generated provided the homomorphism $j_{\overline{U}, bdU}^{n-1}$ is trivial.

We fix $U \in \mathcal{B}'_x$ and a non-zero element $\alpha_x \in H^{n-1}(bd \overline{W}_x; G)$ such that \overline{W}_x is an $(n-1)$ -cohomology membrane for α_x spanned on $bd \overline{W}_x$. Then α_x is not extendable over \overline{W}_x but it is extendable over every closed proper subset of \overline{W}_x . Next, extend α_x to an element $\tilde{\alpha}_x \in H^{n-1}(\overline{W}_x \setminus U; G)$. Obviously, $bdU \subset \overline{W}_x \setminus U$. Hence, the element $\gamma_U = j_{\overline{W}_x \setminus U, bdU}^{n-1}(\tilde{\alpha}_x) \in H^{n-1}(bdU; G)$ is not extendable over \overline{U} (otherwise α_x would be extendable over \overline{W}_x), in particular $\gamma_U \neq 0$. Since U is connected, by Lemma 2.6, \overline{U} is an $(n-1)$ -cohomology membrane for γ_U spanned on bdU .

To prove the second item (ii), let U_0 be an open subset of X with $\overline{U}_0 \subset U$. Since $\gamma_U \in L_U$ and $\gamma_U \neq 0$, $L_U \neq 0$. For any $\gamma \in L_U$ there are two possibilities: either γ is extendable over \overline{U} or it is not extendable over \overline{U} . In both cases γ is extendable over the set $\overline{U} \setminus U_0$. Indeed, this is clear if γ is extendable on \overline{U} . If γ is not extendable over \overline{U} , then \overline{U} is an $(n-1)$ -cohomology membrane for γ spanned on bdU (Lemma 2.6). Consequently, γ is extendable over $\overline{U} \setminus U_0$ because $\overline{U} \setminus U_0$ is a proper subset of \overline{U} containing bdU . Hence, every $\gamma \in L_U$ is extendable over the set $\overline{W}_x \setminus U_0$, which is closed in X and contains bdU in its interior. Therefore, by [4, Theorem 17.4 and Corollary 17.5, p.127], L_U is finitely generated. If $j_{\overline{U}, bdU}^{n-1}(H^{n-1}(\overline{U}; G)) = 0$, then every $\gamma \in H^{n-1}(bdU; G)$ is extendable over $\overline{W}_x \setminus U$, see Lemma 2.5. Hence, $H^{n-1}(bdU; G) \subset L_U$, and item (ii) yields item (iii).

Let \mathcal{B}_x'' be the family of all $U \in \mathcal{B}_x'$ satisfying the following condition: bdU contains a continuum F_U such that $X \setminus F_U$ has exactly two components and F_U is an $(n-1, G)$ -bubble with respect to the module L_U .

Claim 2. \mathcal{B}_x'' is a local base at x .

We fix $W_0 \in \mathcal{B}_x'$ and for every $\delta > 0$ denote by $B(x, \delta)$ the open ball in X with a center x and a radius δ . There exists $\varepsilon_x > 0$ such that $B(x, \delta) \subset W_0$ for all $\delta \leq \varepsilon_x$. Since d is a convex metric, each $B(x, \delta)$ is a connected open set such that $\text{int}(\overline{B(x, \delta)}) = B(x, \delta)$. Because \overline{W}_0 is contractible in W_x , so is $\overline{B(x, \delta)}$. Hence, all $U_\delta = B(x, \delta)$, $\delta \leq \varepsilon_x$, belong to \mathcal{B}_x' . Consequently, by Claim 1, the modules $L_\delta = j_{\overline{W}_x \setminus U_\delta, bdU_\delta}^{n-1}(H^{n-1}(\overline{W}_x \setminus U_\delta; G))$ are finitely generated. Then, by Lemma 2.8, there exists a closed non-empty set $F_\delta \subset bdU_\delta$ with F_δ being an $(n-1; G)$ -bubble with respect to L_δ . Because F_δ is a carrier for any $\gamma \in L_\delta$, Lemma 2.7 yields that each F_δ is a continuum. Let us show that the family $\{F_\delta : \delta \leq \varepsilon_x\}$ is uncountable. Since the function $f: X \rightarrow \mathbb{R}$, $f(y) = d(x, y)$, is continuous and W_0 is connected, $f(W_0)$ is an interval containing $[0, \varepsilon_x]$ and $f^{-1}([0, \varepsilon_x]) = B(x, \varepsilon_x) \subset W_0$. So, $f^{-1}(\delta) = bdU_\delta \neq \emptyset$ for all $\delta \leq \varepsilon_x$. Hence, the family $\{F_\delta : \delta \leq \varepsilon_x\}$ is indeed uncountable and consist of disjoint continua. Moreover, $H^{n-1}(F_\delta; G) \neq 0$ and, according to Lemma 2.4, F_δ separates X . So, each $X \setminus F_\delta$ has at least two components. Then, by [7, Theorem 8], there exists $\delta_0 \leq \varepsilon_x$ such that $X \setminus F_{\delta_0}$ has exactly two components. Therefore, $U_{\delta_0} = B(x, \delta_0) \in \mathcal{B}_x''$ and it is contained in W_0 . This completes the proof of Claim 2.

Now, let \mathcal{B}_x be the subfamily of all $U \in \mathcal{B}_x''$ such that $H^{n-1}(bdU; G) \neq 0$ and both U and $X \setminus \overline{U}$ are connected.

Claim 3. \mathcal{B}_x is a local base at x .

We take an arbitrary neighborhood U_0 of x such that \overline{U}_0 is contractible in W_x and shall construct a member of \mathcal{B}_x contained in U_0 . To this end let $\varepsilon = d(x, X \setminus U_0)$. According to the Effros' theorem [9], there is $\eta > 0$ such that if $y, z \in X$ with $d(y, z) < \eta$, then $h(y) = z$ for some homeomorphism $h: X \rightarrow X$, which is $\varepsilon/2$ -close to the identity on X . Now, choose a connected neighborhood W of x with $\overline{W} \subset B(x, \varepsilon/2)$ and $\text{diam}(\overline{W}) < \eta$. Finally, take $U \in \mathcal{B}_x''$ such that \overline{U} is contractible in W . There exists a continuum $F_U \subset bdU$ such that $X \setminus F_U$ has exactly two components and F_U is an $(n-1, G)$ -bubble with respect to the module $L_U = j_{\overline{W}_x \setminus U, bdU}^{n-1}(H^{n-1}(\overline{W}_x \setminus U; G))$ (see Claim 2). If $F_U = bdU$ we are done, for U is the desired member of \mathcal{B}_x .

Suppose that F_U is a proper subset of bdU . Because F_U is an $(n-1, G)$ -bubble with respect to L_U , $j_{bdU, F_U}^{n-1}(L_U) \neq 0$. Hence, there exists $\gamma \in L_U$ such that $\beta = j_{bdU, F_U}^{n-1}(\gamma) \neq 0$. Because F_U (as a subset of \overline{U}) is contractible in W and \overline{W} (as a subset of \overline{W}_x) is contractible in a proper subset of X , we can apply Lemma 2.4 to conclude that F_U separates \overline{W} . So, $\overline{W} \setminus F_U = V_1 \cup V_2$ for some open, non-empty disjoint subsets $V_1, V_2 \subset \overline{W}$. Since U is a connected subset of $\overline{W} \setminus F_U$, U is contained in one of the sets V_1, V_2 , say $U \subset V_1$. Hence, $F_U \cup \overline{V}_2 \subset \overline{W}_x \setminus U$. Observe that $\gamma \in L_U$ implies γ is extendable over $\overline{W}_x \setminus U$. Consequently, β is also extendable over $\overline{W}_x \setminus U$, in particular

β is extendable over $F_U \cup \overline{V}_2$. On the other hand, F_U (as a subset of \overline{U}) is contractible in \overline{W} , so β is not extendable over \overline{W} (otherwise β would be zero). Thus, since $(F_U \cup \overline{V}_1) \cap (F_U \cup \overline{V}_2) = F_U$, β is not extendable over $F_U \cup \overline{V}_1$. Let $\beta' = j_{F_U, F'}^{n-1}(\beta)$, where $F' = \overline{V}_1 \cap F_U$ (observe that $F' \neq \emptyset$ because \overline{W} is connected). If F' is a proper subset of F_U , then $\beta' = 0$ (recall that $j_{bdU, F'}^{n-1}(\gamma) = \beta'$ and F_U being a carrier for any non-trivial element of $j_{bdU, F_U}^{n-1}(L_U)$ yields $j_{bdU, Q}^{n-1}(L_U) = 0$ for any proper closed subset Q of F_U). So, β' would be extendable over \overline{V}_1 , which yields β is extendable over $F_U \cup \overline{V}_1$, a contradiction. Therefore, $F' = F_U \subset \overline{V}_1$ and β is not extendable over \overline{V}_1 . Consequently, there exists an $(n-1)$ -cohomology membrane $P_\beta \subset \overline{V}_1$ for β spanned on F_U . By Corollary 2.2, $V = P_\beta \setminus F_U$ is a connected open set in X whose boundary, according to Proposition 2.1, is the set $F'' = \overline{X \setminus P_\beta} \cap \overline{P_\beta \setminus F_U} \subset F_U$ (we can apply Proposition 2.1 and Corollary 2.2 because P_β , as a subset of \overline{W}_x , is contractible in a proper subset of X). As above, using that β is not extendable over P_β and $j_{bdU, Q}^{n-1}(L_U) = 0$ for any proper closed subset $Q \subset F_U$, we can show that $F'' = F_U$ and $bd\overline{V} = F_U$. Summarizing the properties of V , we have that \overline{V} is contractible in W_x (because so is \overline{U}_0), $V = \text{int}(\overline{V})$ (because $F_U = bd\overline{V}$) and V is connected. Moreover, since $X \setminus F_U$ is the union of the open disjoint non-empty sets V and $X \setminus P_\beta$ such that V is connected and $X \setminus F_U$ has exactly two components, $X \setminus \overline{V}$ is also connected. Because F_U is an $(n-1, G)$ -bubble with respect to the non-trivial module L_U , $H^{n-1}(bdV; G) \neq 0$. Thus, if V contains x , then V is the desired member of \mathcal{B}_x .

If V does not contain x , we take a point $y \in V$ and a homeomorphism h on X such that $h(y) = x$ and $d(z, h(z)) < \varepsilon$ for all $z \in X$. Such a homeomorphism exists because $\text{diam}(\overline{W}) < \eta$ and $x, y \in \overline{W}$. Then $h(V) \subset U_0$ (this inclusion follows from the choice of ε and the fact that h is ε -close to the identity on X). So, $\overline{h(V)}$ is contractible in W_x . Since the remaining properties from the definition of \mathcal{B}_x are invariant under homeomorphisms, $h(V)$ is the desired member of \mathcal{B}_x , which completes the proof of Claim 3.

The sets $U \in \mathcal{B}_x$ satisfy condition (1) from Theorem 1.1 (according to the definition of \mathcal{B}_x). Next claim completes the proof of Theorem 1.1.

Claim 4. Every $U \in \mathcal{B}_x$ satisfies conditions (2) and (3) from Theorem 1.1.

Recall that each \overline{U} is contractible in the set W_x and \overline{W}_x is contractible in a proper subset of X . Then, by Lemma 2.4, $H^{n-1}(\overline{U}; G) = 0$ because \overline{U} does not separate X . Therefore, every non-trivial element $\gamma \in H^{n-1}(bdU; G)$ is not extendable over \overline{U} . Consequently, according to Lemma 2.6, \overline{U} is an $(n-1)$ -cohomology membrane for γ spanned on bdU . So, U satisfies condition (2).

Since $H^{n-1}(\overline{U}; G) = 0$, the homomorphism $j_{\overline{U}, bdU}^{n-1}$ is trivial. Thus, Lemma 2.5 yields $H^{n-1}(bdU; G) = j_{\overline{W}_x \setminus U, bdU}^{n-1}(H^{n-1}(\overline{W}_x \setminus U; G))$ and, by Claim 1(iii), $H^{n-1}(bdU; G)$ is finitely generated. Suppose there exists a proper closed subset $F \subset bdU$ and a non-trivial element $\alpha \in H^{n-1}(F; G)$. Observe that

α is not extendable over \overline{U} because $H^{n-1}(\overline{U}; G) = 0$. Hence, there is an $(n-1)$ -cohomology membrane $K_\alpha \subset \overline{U}$ for α spanned on F . Because $\overline{U} \setminus F$ is connected (recall that U is a dense connected subset of $\overline{U} \setminus F$) and $K \setminus F$ is both open and closed in $\overline{U} \setminus F$ (by Corollary 2.2), $K_\alpha = \overline{U}$. Finally, according to Proposition 2.1, we have $(K_\alpha \setminus F) \cap \overline{X \setminus K_\alpha} = \emptyset$. On the other hand, any point from $bdU \setminus F$ belongs to $(K_\alpha \setminus F) \cap \overline{X \setminus K_\alpha}$, a contradiction. Therefore, bdU is an $(n-1, G)$ -bubble and U satisfies condition (3). \square

Proof of Theorem 1.2. If $K = X$, Theorem 1.2 is obviously true. Suppose K is a proper closed subset of X , which is an $(n-1)$ -cohomology membrane spanned on A for some $\gamma \in H^{n-1}(A; G)$, but there exists a point $a \in (K \setminus A) \cap \overline{X \setminus K}$. Take a neighborhood $U \in \mathcal{B}_a$ such that $\overline{U} \cap A = \emptyset$. Since $K \setminus U$ is a closed proper subset of K containing A , γ is extendable over $K \setminus U$. So, there exists $\beta \in H^{n-1}(K \setminus U; G)$ with $j_{K \setminus U, A}^{n-1}(\beta) = \gamma$. Since $K \setminus A$ is connected (see Corollary 2.2), $bdU \cap K \neq \emptyset$. Then $\beta_1 = j_{K \setminus U, bdU \cap K}^{n-1}(\beta)$ is a non-zero element of $H^{n-1}(bdU \cap K; G)$ (otherwise β_1 would be extendable over $\overline{U} \cap K$, and hence, γ would be extendable over K). Since $\dim_G bdU \leq n-1$, β_1 is extendable to an element $\tilde{\beta}_1 \in H^{n-1}(bdU; G)$. So, $\tilde{\beta}_1$ is a non-zero element of $H^{n-1}(bdU; G)$ and, by Theorem 1.1(2), \overline{U} is an $(n-1)$ -cohomology membrane for $\tilde{\beta}_1$ spanned on bdU . Then $\overline{U} \cap K \neq \overline{U}$ would yields that $\tilde{\beta}_1$ is extendable over $\overline{U} \cap K$. Hence, γ would be extendable over K , a contradiction. Thus, $\overline{U} \subset K \setminus A$ which contradicts $a \in \overline{X \setminus K}$. Therefore, $(K \setminus A) \cap \overline{X \setminus K} = \emptyset$. \square

Proof of Corollary 1.3. It was shown in [17] and [19] that the cohomology membranes' property from Theorem 1.2 implies the invariance of domain for homogeneous or locally homogeneous ANR -spaces X with $\dim X = n$. Similar arguments provide the proof when $\dim_G X = n$. Take a point $y \in V = f(U)$ and let $x = f^{-1}(y)$. Choose a connected open set $W \in \mathcal{B}_x$ such that $\overline{W} \subset U$. Then \overline{W} is an $(n-1)$ -cohomology membrane for some $\gamma \in H^{n-1}(bdW; G)$ spanned on bdW . Since $f(\overline{W})$ is homeomorphic to \overline{W} , it is an $(n-1)$ -cohomology membrane for $(f^*)^{-1}(\gamma) \in H^{n-1}(f(bdW); G)$ spanned on $f(bdW)$. Then, by Theorem 1.2, $f(\overline{W}) \setminus f(bdW)$ does not intersect $X \setminus f(\overline{W})$. This means that $f(\overline{W}) \setminus f(bdW)$ is an open set in X , which contains y and is contained in V . So, V is also open. \square

4. PROOF OF THEOREM 1.4 AND COROLLARY 1.5

Let \hat{H}_* be the exact homology (see [18], [20]), and \mathbb{Q}, \mathbb{R} denote the groups of rational and the real numbers, respectively. It is well known that for locally compact metric spaces the exact homology is isomorphic with the Steenrod homology. For any abelian group G the homological dimension $h \dim_G Y$ of a compactum Y is the greatest integer m such that $\hat{H}_m(Y, A; G) \neq 0$ for some closed $A \subset Y$ (if there is no such m , then $h \dim_G Y = \infty$). It follows from the exact sequence

$$0 \rightarrow \text{Ext}(H^{m+1}(Y, A), G) \rightarrow \hat{H}_m(Y, A; G) \rightarrow \text{Hom}(H^m(Y, A), G) \rightarrow 0$$

that $h \dim_G Y \leq \dim Y$. Moreover, by [21], $h \dim_G X$ is the greatest m such that the local homology group $\widehat{H}_m(X, X \setminus x; G) = \varinjlim_{x \in U} \widehat{H}_m(X, X \setminus U; G)$ is not trivial for some $x \in X$.

Proof of Theorem 1.4.

(1) \Rightarrow (2). Suppose X is dimensionally full-valued. Then, according to [11], $h \dim_{\mathbb{Z}} X = \dim_{\mathbb{Z}} X = n$. Hence, $\widehat{H}_n(X, X \setminus x) \neq 0$ for some $x \in X$ (the coefficient group \mathbb{Z} in all homology and cohomology groups is suppressed). Because $\dim X = n$, the groups $\widehat{H}_n(X, X \setminus x)$ and $H_n(X, X \setminus x)$ are isomorphic, see [20, Theorem 4]. So, $H_n(X, X \setminus x) \neq 0$.

(2) \Rightarrow (3). Let $H_n(X, X \setminus x) \neq 0$ for some $x \in X$. Then $H_n(X, X \setminus U) \neq 0$ for sufficiently small neighborhoods U of x in X . Since by [20, Theorem 4] the groups $H_n(X, X \setminus U)$ and $\widehat{H}_n(X, X \setminus U)$ are isomorphic, $\widehat{H}_n(X, X \setminus V) \neq 0$ for some neighborhood V of x . On the other hand, $\dim X = n$ implies $H^{n+1}(X, X \setminus V) = 0$. Hence, it follows from the exact sequence

$$\text{Ext}(H^{n+1}(X, X \setminus V), \mathbb{Z}) \rightarrow \widehat{H}_n(X, X \setminus V) \rightarrow \text{Hom}(H^n(X, X \setminus V), \mathbb{Z}) \rightarrow 0$$

that there exists a nontrivial homomorphism from $H^n(X, X \setminus V)$ into \mathbb{Z} . This yields that $H^n(X, X \setminus V)$ contains elements of infinite order. Thus, $H^n(X, X \setminus V) \otimes \mathbb{Q} \neq 0$ and, by the universal coefficients formula, $H^n(X, X \setminus V; \mathbb{Q}) \neq 0$. So, $\dim_{\mathbb{Q}} X = n$. Because X is an ANR, we have the following inequalities $\dim_{\mathbb{Q}} X \leq \dim_{\mathbb{S}^1} X \leq \dim X$, see Example 1.3(1) and Theorem 12.3(2) from [8]. Therefore, $\dim_{\mathbb{S}^1} X = n$.

(3) \Rightarrow (1). Assume $\dim_{\mathbb{S}^1} X = n$. The exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{S}^1 \rightarrow 0$$

implies that $\dim_{\mathbb{S}^1} X \leq \max\{\dim_{\mathbb{R}} X, \dim X - 1\}$, see [8]. Hence, $\dim_{\mathbb{R}} X = n$. According to [11], both the homological and the cohomological dimensions with respect to any field coincide, so we have $h \dim_{\mathbb{R}} X = \dim_{\mathbb{R}} X = n$. Thus, there exist $x \in X$ and a neighborhood U of x in X such that $\widehat{H}_n(X, X \setminus U; \mathbb{R}) \neq 0$. As in the proof of the implication (2) \Rightarrow (3), considering the short exact sequence

$$\text{Ext}(H^{n+1}(X, X \setminus U), \mathbb{Z}) \rightarrow \widehat{H}_n(X, X \setminus U) \rightarrow \text{Hom}(H^n(X, X \setminus U), \mathbb{Z}) \rightarrow 0,$$

we can show that $\dim_{\mathbb{Q}} X = n$. This implies that X is dimensionally full-valued. \square

Proof of Corollary 1.5. Let X be a metric homogeneous ANR compactum with $\dim X = 3$. According to [14, Corollary 2.7], we have $\overline{H}_3(X, X \setminus x) \neq 0$, where $\overline{H}_3(X, X \setminus x)$ denotes the singular homology group. On the other hand, by [15, Lemma 4], the groups $\overline{H}_3(X, X \setminus x)$ and $H_3(X, X \setminus x)$ are isomorphic. Then, Theorem 1.4 yields that X is dimensionally full-valued. \square

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